

# A note on holomorphic extensions

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**Abstract.** *We give a criterium of holomorphy for some type formal power series. This gives a stronger form of a Rothstein's type extension theorem for a particular ring of holomorphic functions.*

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We consider the set  $R \subset \mathbf{C}[[z_1, z_2]]$ ,  $z_1, z_2 \in \mathbf{C}^k \times \mathbf{C}^m$  of formal power series of the form

$$f(z) = f(z_1, z_2) = \sum_n P_n(z_2) z_1^n$$

where  $P_n$  is a polynomial in  $m$  variables of total degree  $\deg P_n \leq C_0 + C_1 \|n\|$ , for some constants  $C_0, C_1 > 0$ . One easily checks that  $R$  is a local sub-ring of  $\mathbf{C}[[z_1, z_2]]$ . For the notion of  $\Gamma$ -capacity, that generalizes the notion of capacity in one complex variable, we refer to [Ro].

**Theorem.** *Let*

$$f(z_1, z_2) = \sum_n P_n(z_2) z_1^n$$

*be a formal power series of the two complex variables  $(z_1, z_2) \in \mathbf{C}^k \times \mathbf{C}^m$ . We assume that  $(P_n)$  is a sequence of polynomials in  $m$  variables of total degree*

$$\deg P_n \leq C_0 + C_1 \|n\| .$$

*We assume that for a set  $K \in \mathbf{C}^m$  of positive  $\Gamma$ -capacity,  $z_2 \in K$  being fixed, the formal power series  $f(z_1, z_2)$  converges.*

*Then for some  $C_2 > 0$ , the formal power series  $f$  defines a holomorphic function in a neighborhood of the axes  $\{z_1 = 0\}$  of the form,*

$$U = \{(z_1, z_2) \in \mathbf{C}^k \times \mathbf{C}^m; \|z_1\| \leq \frac{C_2}{1 + \|z_2\|}\} .$$

Compare with Rothstein's theorem (see [Siu] p.25). Our theorem is motivated and has applications in problems of holomorphic dynamics and small divisors ([PM]) where power series in the ring  $A$  appear naturally.

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**$\Gamma$ -capacity.**

We refer to [Ro]. Let  $K \subset \mathbf{C}^m$ . The  $\Gamma$ -projection of  $K$  on  $\mathbf{C}^{m-1}$  is the set  $\Gamma_m^{m-1}(K)$  of  $z = (z_1, \dots, z_{m-1}) \in \mathbf{C}^{m-1}$  such that

$$K \cap \{(z, w) \in \mathbf{C}^m\}$$

has positive capacity in the complex plane  $\mathbf{C}_z = \{(z, w) \in \mathbf{C}^m\}$ . We define

$$\Gamma_m^1(K) = \Gamma_2^1 \circ \Gamma_3^2 \circ \dots \Gamma_m^{m-1}(K) .$$

Finally, the  $\Gamma$ -capacity is defined as

$$\Gamma\text{-Cap}(K) = \sup_{A \in U(m, \mathbf{C})} \text{Cap } \Gamma_m^1(A(K)) .$$

where  $A$  runs over all unitary transformations of  $\mathbf{C}^m$ .

We have the following lemma ([Ro] Lemma 2.2.8 p.92)

**Lemma.** *Let  $K \subset \mathbf{C}^m$ ,  $K \neq \mathbf{C}^m$  and assume that the intersection of  $K$  with any complex line which is not a subset of  $K$  has inner capacity zero. Then the  $\Gamma$ -capacity of  $K$  is zero.*

Thus we are reduced to prove the theorem for  $m = 1$

**Bernstein lemma.**

We recall (see [Ra] p.156):

**Lemma (Bernstein).** *Let  $K \subset \mathbf{C}$  be a non-polar set, and  $\Omega$  be the component of  $\overline{\mathbf{C}} - K$  containing  $\infty$ .*

*If  $P$  is a polynomial of degree  $n$ , then for  $z \in \mathbf{C}$*

$$|P(z)| \leq \|P\|_{C^0(K)} e^{ng_\Omega(z, \infty)}$$

where  $g_\Omega$  is the Green function of  $\Omega$ .

**Proof of the theorem.**

We are reduced to prove the theorem for  $m = 1$ . For  $z_2 \in K$ , let  $R(z_2)$  be the radius of convergence in  $z_1$  of  $f(z_1, z_2)$ . Let  $K_i = \{z_2 \in K; R(z_2) \geq 1/i\}$ . Since a countable union of polar sets is polar, there is  $K_i$  non-polar. We can take a non-polar sub-compact  $L \subset K_i$  so that there exists  $\rho_0 > 0$  such that for all  $z_2 \in L$

$$\limsup_{\|n\| \rightarrow +\infty} |P_n(z_2)| \rho_0^{-\|n\|} < +\infty .$$

Define

$$\varphi(z_2) = \limsup_{\|n\| \rightarrow +\infty} |P_n(z_2)| \rho_0^{-\|n\|} .$$

The function  $\varphi$  is lower semi-continuous, and

$$L = \bigcup_{p \geq 1} L_p$$

where  $L_p = \{z \in L; \varphi(z_2) \leq p\}$  is closed. By Baire theorem for some  $p$ ,  $L_p$  has non-empty interior (with respect to  $L$ ), thus some  $L_p$  has positive capacity. Finally we found a compact set  $C = L_p$  of positive capacity such that there exists  $\rho_1 > 0$  such that for any  $z_2 \in C$  and  $n$ ,

$$|P_n(z_2)| \leq \rho_1^{\|n\|}.$$

Now using Bernstein lemma we conclude that for any  $z_2 \in \mathbf{C}$ , for all  $n$ ,

$$|P_n(z_2)| \leq \rho_1^{\|n\|} e^{(C_0 + C_1 \|n\|)g_\Omega(z_2, \infty)}.$$

Finally using the asymptotic

$$g_\Omega(z_2, \infty) = \log |z_2| + \mathcal{O}(1)$$

we obtain the extension to the desired domain.

**Remark.**

1. We can improve on the domain of extension if we control the growth of the degrees of the polynomials  $(P_n)$ . For instance, the same proof shows that if

$$\limsup_n \frac{1}{\|n\|} \deg P_n = 0$$

we have a holomorphic extension to a domain

$$U = \{(z_1, z_2) \in \mathbf{C}^k \times \mathbf{C}^m; \|z_1\| \leq C\}$$

for some  $C > 0$ .

2. As N. Sibony has pointed out recently to me, the condition positive  $\Gamma$ -capacity in the theorem can be replaced by non-pluri-polar set (which is stronger and more natural) using the definition of capacity in higher dimension and the techniques of [Al] and [Si]. Essentially one writes down a general Bernstein lemma in higher dimension (similar to lemma 6.5 in [Al]) and use it as we do in dimension 1. The main dynamical applications, where we seek for generic conditions, work as well with the version with  $\Gamma$ -capacity. For this first version we content ourselves with the above statement.

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